Math 847 Qualifying Examination August 2019

Instructions: You must show all necessary work to get full credits. You can use a 3×5 index card. You can not use your book, cell phone, computer, or other notes. Read all problems through once carefully before beginning work. Partial credit will only be given for progress toward a correct solution.

Notation: $\mathbf{R}^n$ denotes the standard Euclidean space with $|x| = \sqrt{x_1^2 + ... + x_n^2}$ for $x = (x_1, ..., x_n) \in \mathbf{R}^n$. $\Delta u(x) = \sum_{j=1}^n u_{x_j x_j}$ stands for the Laplace operator in $\mathbf{R}^n$. You can use ω_n to denote the surface area for $S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\}$ if necessary.

Problem 1. Consider the first order equation $u_t(x,t) + 2tu_x(x,t) = 4u,\ x \in \mathbf{R},\ t \in \mathbf{R}$.

(a) Find a solution $u(x,t)$ with initial data $u(x,0) = x+1$, $x \in \mathbf{R}$. Check your answer by direct differentiation.

(b) What is the characteristic curve starting from $x_0 = 0, t_0 = 0$? Can you find a C^1 solution $u(x,t)$ such that $u(s^2, s) = 2e^{4s}$ for all $s \in \mathbf{R}$? Explain your answer.

Problem 2. Let $\Omega = \{x = (x_1, x_2) \in \mathbf{R}^2 \mid |x| < 1\}$, and consider the following boundary value problem

$$\begin{cases} \triangle u = 0, & x \in \Omega, \\ u(x_1, x_2) = 3x_1 - x_1^2 + 2x_2^2, & (x_1, x_2) \in \partial\Omega. \end{cases}$$

(a) Prove that the boundary value problem has one and only one solution.

(b) What is $u(0,0)$?

(c) Can you find a point $x \in \Omega$ such that $u(x) = 3$? Explain your answer.

Problem 3. Consider the following initial value problem

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) + t^2 & x \in \mathbf{R},\ t > 0, \\ u(x,0) = 0, & x \in \mathbf{R}, \\ u_t(x,0) = \frac{1}{1+x^2}, & x \in \mathbf{R}. \end{cases}$$

(a) Find a solution to the initial value problem.

(b) Can you find a different $C^2(\mathbf{R} \times \mathbf{R}^+)$ solution? Briefly explain your answer.

Problem4. Let $f(x)$ be a continuous function on $\mathbf{R}^n$, $2 \geq f(x) > 0$ for all $x \in \mathbf{R}^n$, and $\int_{\mathbf{R}^n} f(x)dx = 1$. Consider the initial value problem

$$\begin{cases} u_t(x,t) - \triangle u(x,t) + 4u = 0, & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = f(x), & x \in \mathbf{R}^n. \end{cases}$$

(a) Prove that there exists a solution $u(x,t)$ such that $0 < u(x,t) < e^{-4t} \min\{2, \frac{1}{(4\pi t)^{n/2}}\}$ for all $x \in \mathbf{R}^n$ and $t > 0$.

(b) Can you find another different solution to the initial value problem? Can you find another different solution $u(x,t)$ such that $|u(x,t)| \leq 2$ to the initial value problem? Explain your answer.

Problem 5. Let $f(x) \in C_0^\infty(\mathbf{R}^n)$ such that $f(x) = 1$ for $|x| \leq 2$, $f(x) \geq 0$ for $2 < |x| < 4$, and $f(x) = 0$ for all $|x| \geq 4$. Consider the second order differential equation

$$\Delta u(x) = f(x), \quad x \in \mathbf{R}^n.$$

(a) Using the fundamental solution for Δ, give an integral expression for a solution to this equation for $n = 3$.

(b) Prove that there are infinitely many solutions to this equation with $u(0) = 0$ when $n = 3$.

(c) prove that there is one and only one solution $u(x)$ such that $u(x)$ is bounded and $u(0) = 0$ when $n = 3$.

(d) Let $g(r) = \int_{|y|=1} u(ry)dS_y$ for any solution to the original equation. Prove that $g(r)$ is a strictly increasing and bounded function for $r \in (0, \infty)$ (even for unbounded solutions) when $n = 3$.

Problem 6. Consider the following initial and boundary value problem for a nonlinear heat equation

$$\begin{cases} u_t = 4u_{xx} + u_{yy} - 2u_x^2 + u_y, & x^2 + y^2 < 1, t > 0, \\ u(x,y,0) = (1 - x^2 - y^2)^4, & x^2 + y^2 < 1, \\ u(x,y,t) = 0, & x^2 + y^2 = 1, t \geq 0. \end{cases}$$

(a) If $u(x,t)$ is a smooth solution of the problem, prove that $0 \leq u(x,y,t) \leq 1$ for all $x^2+y^2 < 1$ and $t > 0$.

(b) Prove that the problem has at most one solution.

Problem 7. Consider the initial value problem for the wave equation in $\mathbb{R}^3 \times \mathbb{R}$

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}, \quad u(x,y,z,0) = \frac{1}{(1 + x^2 + y^2 + z^2)^2}, \quad u_t(x,y,z,0) = 0.$$

(a) Find a solution of the form $u(x,y,z,t) = V(r,t)$ with $r = \sqrt{x^2 + y^2 + z^2}$.

(b) Can you find two different C^2 solutions to the original initial value problem? Briefly explain your answer.

(c) Prove that $E_1(t) = \int_{R^3} u(x,y,z,t)dxdydz \equiv E_1(0)$.